Euclidean Spacetime

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Includes new figures and minor edits. Original text-only Backreaction version is at:
https://backreaction.blogspot.com/2020/08/do-we-really-travel-through-time-with.html?showComment=1599592887181#c3720712902561121259

Sabine,

Thanks for an excellent and informative video! This belated comment is on how minor algebraic manipulations that convert Minkowski spacetime into Euclidean form can make the relationship between time and distance particularly vivid.

If the Deltas are implicit your spacetime equation was:

$$s = \sqrt{-x^2 - y^2 - z^2 + (ct)^2}$$

Squaring both sides gives:

$$s^2 = -x^2 - y^2 - z^2 + (ct)^2$$

Shuffling terms doesn't change the meaning, so here is a positive-only version of the above sum-of-squares (SOS) equation:

$$(ct)^2 = s^2 + x^2 + y^2 + z^2$$

A nifty feature of positive-only SOS equations is that both sides of the equation can be interpreted as boxes — distorted cubes. These two boxes share a body diagonal and are embedded in a Euclidean space of one less dimension than the total number of squared terms. Another useful feature is that any group of squares on one side can be replaced by a single squared term that represents a "box diagonal", more commonly known as a *vector*. Thus:

$$r = \sqrt{x^2 + y^2 + z^2}$$

or

$$r^2 = x^2 + y^2 + z^2$$

Substituting back gives:

$$(ct)^2 = s^2 + r^2$$

Sabine noted that for no motion, $s/t = c$, or $s = ct$. However, for the full case of motion the time will be different from t, so let's call this more general case $s = ct'$. This gives:

$$(ct)^2 = (ct')^2 + r^2$$

Dividing by c^2 gives time:

$$t^2 = t'^2 + (r/c)^2$$

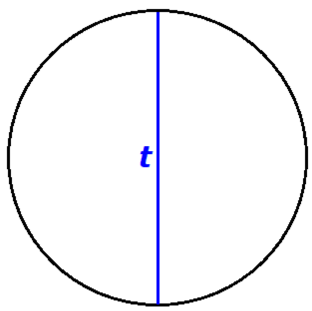
… which is the Pythagorean equation for a right triangle.

Notice that even in its 4D form this equation uses an all-positive signature Euclidean space, rather than the mixed signature of Minkowski space.

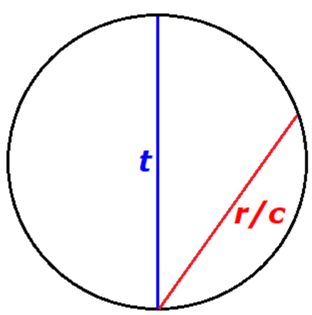
The biggest advantage of Euclidean spacetime is that it transforms the often-obscure relationships of Minkowski SR into simple, easy-to-understand right triangles.

Here's an example:

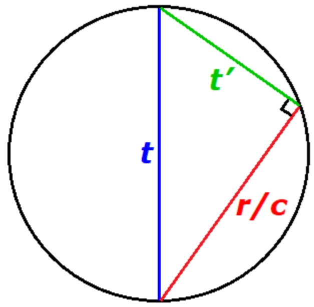
Draw a circle, then divide it in half with a vertical line. Label this vertical diameter line $\boldsymbol{t}$, the time elapsed by a clock in the rest frame.



Next, start at the bottom of $\boldsymbol{t}$ and draw a line to any point on the side of the circle. Label this new line $\boldsymbol{r/c}$. It represents *how far some clock traveled* during the rest clock interval $\boldsymbol{t}$.



Finally, connect the top of t to the top of r/c, and label this last line t'. Geometry requires that this new line will form a $90°$ angle with r/c, thus completing the right triangle. This last line represents how many ticks the moving clock will manage to get in before the clock-at-rest completes t ticks.



Oddly, that's it! If your other clock just stays put, as in Sabine's example, the first side becomes a dot, and the second side $t' = t$. The faster the clock moves, the longer the distance line becomes, and the fewer ticks the moving clock can get in — a very literal, in-your-face tradeoff between distance and time. Finally, if the object gets very, very close to the speed of light, the distance-traveled side gobbles up all of the available t diameter, and the moving clock has no ticks at all.

This right-triangle relationship doesn't end with time. Since such triangles amount to a different way of representing the Lorentz factor, all of the major features of Minkowski SR also have right triangle representations in Euclidean SR.

So my point in all of this?

Well, first, I just wanted to show that switching over to Euclidean SR makes the relationship between space and time as point-blank as possible, with the distance vector of the moving clock essentially taking over the time vector of the rest frame, converting it from ticks to travel.

But the other point is to say that, as in all areas of math, choice of representation matters. Minkowski space does a better job of keeping our usual intuitions of space and time as pre-existing, whereas Euclidean spacetime is much more relational, requiring both frames to "talk" about what direction means what. But both represent the same underlying relationships, and thus add complementary value to understanding SR.
